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Singular integral operators in Morrey spaces and interior regularity of solutions to systems of linear PDE's

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Abstract We obtain boundedness in Morrey spaces of singular integral operators with Calderón-Zygmund type kernel of mixed homogeneity. These estimates are used for the study of the interior regularity of the solutions of linear elliptic/parabolic systems. The proved Poincaré-type inequality permits to describe the Hölder, Morrey, and BMO regularity of the lower-order derivatives of the solutions.

Keywords Singular integral operators · A'priori estimates · Morrey spaces · Elliptic and parabolic systems · *VMO* coefficients · Hölder regularity

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1 Singular integral estimates in Morrey spaces

We are interested in continuity in Morrey spaces of the following integral operators

$$\begin{aligned} \mathfrak{K}f(x) &= P.V. \int_{\mathbb{R}^n} k(x; x-y) f(y) \mathrm{d}y \\ \mathfrak{C}[a, f](x) &= P.V. \int_{\mathbb{R}^n} k(x; x-y)[a(y)-a(x)] f(y) \mathrm{d}y. \end{aligned} \tag{1}$$

The kernel $k(x, \xi)$ is a singular one, satisfying Calderón–Zygmund type conditions. Precisely, let $\alpha_1, \dots, \alpha_n$ be real numbers, $\alpha_i \geq 1$, $\alpha = \sum_{i=1}^n \alpha_i$ and set $\mathbb{S}^{n-1}$ for the unit sphere in $\mathbb{R}^n$.

Definition 1 *A function* $k(x; \xi)\colon\ \mathbb{R}^n \times \{\mathbb{R}^n \setminus \{0\}\} \to \mathbb{R}$ *is a* variable kernel of mixed homogeneity *if:*

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(*i*) $k(x;\cdot)$ *is a Calderón-Zygmund type kernel for almost all fixed* $x \in \mathbb{R}^n$, *i.e.*
  (i_a) $k(x,\cdot) \in C^\infty(\mathbb{R}^n \setminus \{0\})$;
  (i_b) $k(x, \mu^{\alpha_1}\xi_1, \ldots, \mu^{\alpha_n}\xi_n) = \mu^{-\alpha}k(x,\xi)$ *for each* $\mu > 0$;
  (i_c) $\int_{\mathbb{S}^{n-1}} |k(x,\xi)|d\sigma_\xi < \infty$ *and* $\int_{\mathbb{S}^{n-1}} k(x,\xi)d\sigma_\xi = 0$.
(*ii*) $\sup_{\xi\in\mathbb{S}^{n-1}} \left|D_\xi^\beta k(x;\xi)\right| \le C(\beta)$ *for every multiindex* β, *independently of* x.

In the special case $\alpha_i = 1$ and thus $\alpha = n$, Definition 1 gives rise to the variable Calderón-Zygmund kernel (cf. [2,3,6,18]). The mixed homogeneity condition suggests to endow $\mathbb{R}^n$ with a metric that takes into account (i_b). Thus, following Fabes and Riviére [10], the function $F(x,\rho) = \sum_{i=1}^n x_i^2/\rho^{2\alpha_i}$, considered for a fixed $x \in \mathbb{R}^n$, is a decreasing one in $\rho > 0$ and the equation $F(x,\rho) = 1$ possesses a unique solution $\rho(x)$. It is a simple matter to check that $\rho(x-y)$ defines a distance between any two points $x, y \in \mathbb{R}^n$ ([10,Remark 1]). Further, $\overline{x} := \frac{x}{\rho(x)} := \left(\frac{x_1}{\rho(x)^{\alpha_1}}, \ldots, \frac{x_n}{\rho(x)^{\alpha_n}}\right) \in \mathbb{S}^{n-1}$. The balls with respect to $\rho(x)$, centered at the origin and of radius r are the ellipsoids

$$\mathcal{E}_r(0) = \left\{x \in \mathbb{R}^n : \quad \frac{x_1^2}{r^{2\alpha_1}} + \cdots + \frac{x_n^2}{r^{2\alpha_n}} < 1\right\}, \quad \text{meas}\,(\mathcal{E}_r) \sim r^\alpha$$

and meas $(\mathcal{E}_r)$ stands for the Lebesgue measure. For the sake of completeness the definitions of the spaces we are going to use are given with respect to the metric $\rho(x)$.

Definition 2 *For measurable and locally integrable function f:* $\mathbb{R}^n \to \mathbb{R}$ *set*

$$\eta_f(R) = \sup_{r\le R} \frac{1}{|\mathcal{E}_r|} \int_{\mathcal{E}_r} |f(y) - f_{\mathcal{E}_r}|\mathrm{d}y \quad \textit{for every } R > 0,$$

where $\mathcal{E}_r$ *is any ellipsoid in* $\mathbb{R}^n$ *of radius r, and* $f_{\mathcal{E}_r} = |\mathcal{E}_r|^{-1}\int_{\mathcal{E}_r} f(y)dy$. *Then:*

- $f \in BMO$ (bounded mean oscillation, [15]) *if* $\|f\|_* := \sup_R \eta_f(R) < +\infty$. $\|f\|_*$ *is a norm in* BMO *modulo constant functions under which* BMO *is a Banach space.*
- $f \in VMO$ (vanishing mean oscillation, [22]) *if* $f \in BMO$ *and* $\lim_{R\to 0}\eta_f(R) = 0$. *The quantity* $\eta_f(R)$ *is referred to as a* VMO*-modulus of* f.

For a given domain $\Omega \subset \mathbb{R}^n$, *the spaces* $BMO(\Omega)$ *and* $VMO(\Omega)$ *are defined in the same manner, just taking* $\mathcal{E}_r \cap \Omega$ *instead of* $\mathcal{E}_r$ *above.*

Definition 3 *A measurable function* $f \in L^p(\mathbb{R}^n)$, $p \in (1,+\infty)$, *belongs to the Morrey space* $L^{p,\lambda}(\mathbb{R}^n)$ *with* $\lambda \in (0,\alpha)$, *if*

$$\|f\|_{p,\lambda} = \left(\sup_{r>0} \frac{1}{r^\lambda} \int_{\mathcal{E}_r} |f(y)|^p \mathrm{d}y\right)^{1/p} < \infty, \tag{2}$$

where $\mathcal{E}_r$ *stands for any ellipsoid of radius r. Similarly, the space* $L^{p,\lambda}(\Omega)$ *and the norm* $\|f\|_{p,\lambda,\Omega}$ *are defined by taking* $\mathcal{E}_r \cap \Omega$ *in* (2).

Let $f \in L^{p,\lambda}$ and $a \in BMO$. For $\varepsilon > 0$ define the operators $\mathfrak{K}_\varepsilon f$ and $\mathfrak{C}_\varepsilon[a,f]$ by

$$\mathfrak{K}_\varepsilon f(x) := \int_{\rho(x-y)>\varepsilon} k(x;x-y)f(y)\mathrm{d}y, \quad \mathfrak{C}_\varepsilon[a,f](x) := \mathfrak{K}_\varepsilon(af)(x) - a(x)\mathfrak{K}_\varepsilon f(x).$$

Our main result ensures existence and boundedness of the integrals (1).

Theorem 4 *For any $f \in L^{p,\lambda}$, $p \in (1, +\infty)$, $\lambda \in (0, \alpha)$ and $a \in BMO$ we have*

$$\begin{aligned} \mathfrak{K}f(x) &= \lim_{\varepsilon\to 0} \mathfrak{K}_\varepsilon f(x), \quad \mathfrak{C}[a, f](x) = \lim_{\varepsilon\to 0} \mathfrak{C}_\varepsilon[a, f](x), \\ \|\mathfrak{K}f\|_{p,\lambda} &\le C\|f\|_{p,\lambda}, \quad \|\mathfrak{C}[a, f]\|_{p,\lambda} \le C\|a\|_*\|f\|_{p,\lambda}. \end{aligned} \tag{3}$$

Corollary 5 *Let Ω be an open subset of $\mathbb{R}^n$ and $k(x;\xi) : \Omega\times(\mathbb{R}^n\setminus\{0\}) \to \mathbb{R}$, $a \in BMO(\Omega)$. Then, for any $f \in L^{p,\lambda}(\Omega)$ and almost all $x \in \Omega$ the integrals $\mathfrak{K}f$, $\mathfrak{C}[a, f] \in L^{p,\lambda}(\Omega)$ and there is a constant $C = C(n, p, \lambda, \alpha, \Omega, k)$ such that*

$$\|\mathfrak{K}f\|_{p,\lambda,\Omega} \le C\|f\|_{p,\lambda,\Omega}, \quad \|\mathfrak{C}[a, f]\|_{p,\lambda,\Omega} \le C\|a\|_*\|f\|_{p,\lambda,\Omega}. \tag{4}$$

Corollary 6 *In addition to the assumptions of Corollary 5, let $a \in VMO(\Omega)$ with VMO-modulus η_a. Then, for each $\varepsilon > 0$ there is $r_0 = (\varepsilon, \eta_a) > 0$ such that for any $r \in (0, r_0)$ and any ellipsoid $\mathcal{E}_r \subset \Omega$ one has*

$$\|\mathfrak{C}[a, f]\|_{p,\lambda,\mathcal{E}_r} \le C\varepsilon\|f\|_{p,\lambda,\mathcal{E}_r} \quad \forall f \in L^{p,\lambda}(\mathcal{E}_r). \tag{5}$$

2 Interior regularity for parabolic systems with discontinuous data

Let Ω be a domain in $\mathbb{R}^n$, $n \ge 2$ and define $Q = \Omega \times (0, T)$ with $T > 0$. We consider the following linear system of order $2b$, $b \ge 1$

$$\mathfrak{L}\mathbf{u} := D_t\mathbf{u}(x, t) - \sum_{|\alpha|=2b} \mathbf{A}_\alpha(x, t)D^\alpha\mathbf{u}(x, t) = \mathbf{f}(x, t) \tag{6}$$

for the unknown vector-valued function $\mathbf{u}$: $Q \to \mathbb{R}^m$ given by the transpose $\mathbf{u}(x, t) = \big(u_1(x, t), \dots, u_m(x, t)\big)^{\mathrm{T}}$, $\mathbf{f} = (f_1, \dots, f_m)^{\mathrm{T}}$, and where $\mathbf{A}_\alpha(x, t)$ stands for the $m \times m$ matrix $\left\{a_\alpha^{kj}(x, t)\right\}_{k,j=1}^m$ of the measurable coefficients $a_\alpha^{kj} : Q \to \mathbb{R}$. Here, $\alpha = (\alpha_1, \dots, \alpha_n)$ is a multiindex of length $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $D_t := \partial/\partial t$ and $D^\alpha \equiv D_x^\alpha := D_1^{\alpha_1} \dots D_n^{\alpha_n}$ with $D_i := \partial/\partial x_i$. Further, $D^\alpha\mathbf{u} = \big(D^\alpha u_1, \dots, D^\alpha u_m\big)^{\mathrm{T}}$ and $D^s\mathbf{u}$ substitutes *any* derivative $D^\alpha\mathbf{u}$ with $|\alpha| = s \in \mathbb{N}$.

We assume that the system (6) is uniformly parabolic in the sense of Petrovskii (see [8,9,12,17,24]). Namely, the p-roots of the m-degree polynomial

$$\det\left\{p\,\mathbf{Id}_m - \sum_{|\alpha|=2b} \mathbf{A}_\alpha(x, t)(i\xi)^\alpha\right\} = 0 \quad (i = \sqrt{-1}) \tag{7}$$

satisfy, for some $\delta > 0$ and all $s = 1, \dots, m$, the inequality

$$\operatorname{Re} p_s(x, t, \xi) \le -\delta|\xi|^{2b} \quad \text{for a.a. } (x, t) \in Q, \ \forall\xi \in \mathbb{R}^n. \tag{8}$$

Here $\mathbf{Id}_m$ is the identity $m \times m$ matrix, $\xi^\alpha := \xi_1^{\alpha_1}\xi_2^{\alpha_2} \cdots \xi_n^{\alpha_n}$ and $|\cdot|$ indicates the Euclidean norm in $\mathbb{R}^n$. For fixed $(x, t) \in Q$ and $\xi \in \mathbb{R}^n$, $p_s(x, t, \xi)$ are nothing else than the eigenvalues of the $m \times m$ matrix $(-1)^b \sum_{|\alpha|=2b} \mathbf{A}_\alpha(x, t)\xi^\alpha$, and the parabolicity condition (8) means these have negative real part.

Our goal is to obtain interior Hölder regularity of the strong solutions to (6) as a byproduct of *a priori* estimates in Sobolev and Sobolev–Morrey spaces. Let us recall the definitions of these functional classes.

Definition 7 The parabolic Sobolev space $W_p^{2b,1}(Q)$, $p \in (1,+\infty)$, is the collection of $L^p(Q)$ functions $u\colon Q \to \mathbb{R}$ all of which distribution derivatives $D_t u$ and $D_x^\alpha u$ with $|\alpha| \le 2b$, belong to $L^p(Q)$. The norm in $W_p^{2b,1}(Q)$ is

$$\|u\|_{W_p^{2b,1}(Q)} := \|D_t u\|_{p;Q} + \sum_{s=0}^{2b} \|D^s u\|_{p;Q}, \quad \|\mathbf{u}\|_{p;Q} := \sum_{k=1}^{m} \|u_k\|_{p;Q},$$

When dealing with localized versions of $W_p^{2b,1}$ we always mean *local in spatial variables x and global in time,* that is, $u \in W_{p,\mathrm{loc}}^{2b,1}(Q)$ if $u \in W_p^{2b,1}(\Omega' \times (0,T))$ for any $\Omega' \Subset \Omega$.

Let $p \in (1,+\infty)$ and $\lambda \in (0, n+2b)$. The Sobolev–Morrey space $W_{p,\lambda}^{2b,1}(Q)$ consists of all functions $u \in W_p^{2b,1}(Q)$ with generalized derivatives $D_t u$ and $D_x^\alpha u$, $|\alpha| \le 2b$, belonging to $L^{p,\lambda}(Q)$ and the norm is given by

$$\|u\|_{W_{p,\lambda}^{2b,1}(Q)} := \|D_t u\|_{p,\lambda;Q} + \sum_{s=0}^{2b} \sum_{|\alpha|=s} \|D_x^\alpha u\|_{p,\lambda;Q}.$$

Endow $\mathbb{R}^{n+1} = \mathbb{R}_x^n \times \mathbb{R}_t$ with the *parabolic* metric $\varrho(x,t) = \max\{|x|, |t|^{1/2b}\}$. We shall employ the system of *parabolic cylinders*

$$\mathcal{C}_r := B_r(x_0) \times (t_0 - r^{2b}, t_0), \quad B_r(x_0) := \left\{x \in \mathbb{R}^n\colon \quad |x - x_0| < r\right\} \tag{9}$$

with meas $(\mathcal{C}_r)$ comparable to r^{n+2b}. It is obvious that the metric $\varrho(x,t)\}$ is equivalent to $\rho(x,t)$. In fact, for any ellipsoid $\mathcal{E}$ there exist cylinders $\underline{\mathcal{C}}$ and $\overline{\mathcal{C}}$ of measures equivalent to meas $(\mathcal{E})$ and such that $\underline{\mathcal{C}} \subset \mathcal{E} \subset \overline{\mathcal{C}}$. Our main results are as follows.

Theorem 8 *Suppose* (8)*,* $1 < q \le p < +\infty$*,* $a_\alpha^{kj} \in VMO \cap L^\infty(Q)$*,* $\mathbf{f} \in L_{\mathrm{loc}}^p(Q)$ *and let* $\mathbf{u} \in W_{q,\mathrm{loc}}^{2b,1}(Q)$ *be a strong solution of* (6) *such that* $\mathbf{u}(x,0) = \mathbf{0}$.

Then the operator $\mathfrak{L}$ improves integrability, *that is,* $\mathbf{u} \in W_{p,\mathrm{loc}}^{2b,1}(Q)$*, and for any* $Q' = \Omega' \times (0,T)$*,* $Q'' = \Omega'' \times (0,T)$*,* $\Omega' \Subset \Omega'' \Subset \Omega$*, there is a constant C depending on n, p, m, b,* δ*,* $\|a_\alpha^{kj}\|_{\infty;Q}$*,* $\eta_{a_\alpha^{kj}}$ *and* dist $(\Omega', \partial\Omega'')$ *such that*

$$\|\mathbf{u}\|_{W_p^{2b,1}(Q')} \le C\left(\|\mathbf{f}\|_{p;Q''} + \|\mathbf{u}\|_{p;Q''}\right). \tag{10}$$

Since $W_p^{2b,1}(Q')$ is contained into the Besov space $B_{\infty,\infty}^{\sigma,\sigma/2b}(Q')$ with $\sigma = 2b - \frac{n+2b}{p} > 0$ and $B_{\infty,\infty}^{\sigma,\sigma/2b}(Q')$ coincides with the Hölder space $C^{\sigma,\sigma/2b}(Q')$ for *non-integer* σ (see [13,14,17,25, Theorems 2.5, 2.7]), we get

Corollary 9 *In addition to the hypotheses of Theorem* 8*, suppose* $p > \frac{n+2b}{2b}$*. Then* $\mathbf{u} \in L^\infty(Q')$ *and there is a constant C such that*

$$\|\mathbf{u}\|_{\infty;Q'} \le C\left(\|\mathbf{f}\|_{p;Q''} + \|\mathbf{u}\|_{p;Q''}\right).$$

Moreover, the x-derivatives of $\mathbf{u}$ *are Hölder continuous for large values of p. Precisely,*

- *if* $p \in \left(\frac{n+2b}{2b-s}, \frac{n+2b}{2b-s-1}\right)$ *for a fixed* $s \in \{0,1,\ldots,2b-2\}$ *then* $D^s\mathbf{u} \in C^{\sigma_s,\sigma_s/2b}(Q')$ *with* $\sigma_s = 2b - s - \frac{n+2b}{p}$;
- *if* $p \in (n+2b, +\infty)$ *then* $D^{2b-1}\mathbf{u} \in C^{\sigma_{2b-1},\sigma_{2b-1}/2b}(Q')$ *with* $\sigma_{2b-1} = 1 - \frac{n+2b}{p}$

and in all cases

$$\sup_{\substack{(x,t)\neq(x',t')\\ (x,t),\ (x',t')\in Q'}} \frac{\left|D^s\mathbf{u}(x,t)-D^s\mathbf{u}(x',t')\right|}{\left(|x-x'|+|t-t'|^{1/2b}\right)^{\sigma_s}} \leq C\left(\|\mathbf{f}\|_{p;Q''}+\|\mathbf{u}\|_{p;Q''}\right)$$

for $s\in\{0,1,\ldots,2b-1\}$.

Our next result provides improving-of-integrability property of $\mathfrak{L}$ and *a priori* estimates in Sobolev–Morrey spaces for solutions of (6) with Morrey right-hand side.

Theorem 10 *Suppose* (8), $1<q\leq p<+\infty$, $\lambda\in(0,n+2b)$, $a^{kj}_\alpha\in VMO\cap L^\infty(Q)$, $\mathbf{f}\in L^{p,\lambda}_{\rm loc}(Q)$ *and let* $\mathbf{u}\in W^{2b,1}_{q,\rm loc}(Q)$ *be a strong solution of* (6) *such that* $\mathbf{u}(x,0)=\mathbf{0}$. *Then* $\mathbf{u}\in W^{2b,1}_{p,\lambda,\rm loc}(Q)$ *and*

$$\|\mathbf{u}\|_{W^{2b,1}_{p,\lambda}(Q')}\leq C\left(\|\mathbf{f}\|_{p,\lambda;Q''}+\|\mathbf{u}\|_{p,\lambda;Q''}\right) \tag{11}$$

with a constant C depending on the quantities listed in Theorem 8 and on λ *in addition.*

As consequence of (11) we obtain precise characterization of Morrey, BMO and Hölder regularity of the derivatives $D^s\mathbf{u}$ with $s\in\{0,1,\ldots,2b-1\}$. Precisely,

Corollary 11 *Under the hypotheses of Theorem 10 fix an* $s\in\{0,1,\ldots,2b-1\}$. *Then there is a constant C such that*

- *if* $p\in\left(1,\frac{n+2b-\lambda}{2b-s}\right)$ *then* $D^s\mathbf{u}\in L^{p,(2b-s)p+\lambda}(Q')$ *and*
$$\|D^s\mathbf{u}\|_{p,(2b-s)p+\lambda;Q'}\leq C\left(\|\mathbf{f}\|_{p,\lambda;Q''}+\|\mathbf{u}\|_{p,\lambda;Q''}\right);$$
- *if* $p=\frac{n+2b-\lambda}{2b-s}$ *then* $D^s\mathbf{u}\in BMO(Q')$ *and*
$$\|D^s\mathbf{u}\|_{*;Q'}\leq C\left(\|\mathbf{f}\|_{p,\lambda;Q''}+\|\mathbf{u}\|_{p,\lambda;Q''}\right);$$
- *if* $p\in\left(\frac{n+2b-\lambda}{2b-s},\frac{n+2b-\lambda}{2b-s-1}\right)$[1] *then* $D^s\mathbf{u}\in C^{\sigma_s,\sigma_s/2b}(Q')$ *with* $\sigma_s=2b-s-\frac{n+2b-\lambda}{p}$ *and*
$$\sup_{\substack{(x,t)\neq(x',t')\\ (x,t),\ (x',t')\in Q'}} \frac{\left|D^s\mathbf{u}(x,t)-D^s\mathbf{u}(x',t')\right|}{\left(|x-x'|+|t-t'|^{1/2b}\right)^{\sigma_s}} \leq C\left(\|\mathbf{f}\|_{p,\lambda;Q''}+\|\mathbf{u}\|_{p,\lambda;Q''}\right).$$

A simple geometric interpretation of the results of Corollaries 9 and 11 is proposed on Fig. 1. A typical situation is considered for the couple (p,λ) lying in the semistrip $\{(p,\lambda)\colon\ p>1, 0<\lambda<n+2b\}$ and $s\in\{0,1,\ldots,2b-1\}$. The points B_s on the p-axis are simply $\left(\frac{n+2b}{2b-s},0\right)$, $B=(1,0)$, and $A_s=(1,n+s)$ is the intersection of the line $\{p=1\}$ with the line passing through $(0,n+2b)$ and B_s. If (p,λ) belongs to the open right triangle $\triangle BB_sA_s$ then $D^s\mathbf{u}\in L^{p,(2b-s)p+\lambda}(Q')$. In particular, $(p,\lambda)\in\triangle BB_0A_0$ yields $\mathbf{u}\in L^{p,2bp+\lambda}(Q')$ whereas $\mathbf{u}\in BMO(Q')$ if (p,λ) lies on the open line segment (A_0,B_0). Let $s\in\{0,\ldots,2b-2\}$ and take (p,λ) in the interior of $R_s:=B_sB_{s+1}A_{s+1}A_s$. Then the spatial derivatives $D^s\mathbf{u}$ are Hölder continuous with exponent σ_s given by Corollary 11, while $D^{s+1}\mathbf{u}\in L^{p,(2b-s-1)p+\lambda}(Q')$. Moreover, σ_s is the length $|CA_s|$ of the segment (C,A_s) where $C=C(p,\lambda)$ is the intersection of the vertical line $\{p=1\}$ with the line connecting the points (p,λ) and $(0,n+2b)$. If $(p,\lambda)\in(A_s,B_s)$ we have $D^s\mathbf{u}\in BMO$,

[1] In the case $s=2b-1$ this inclusion rewrites naturally as $p\in(n+2b-\lambda,+\infty)$.

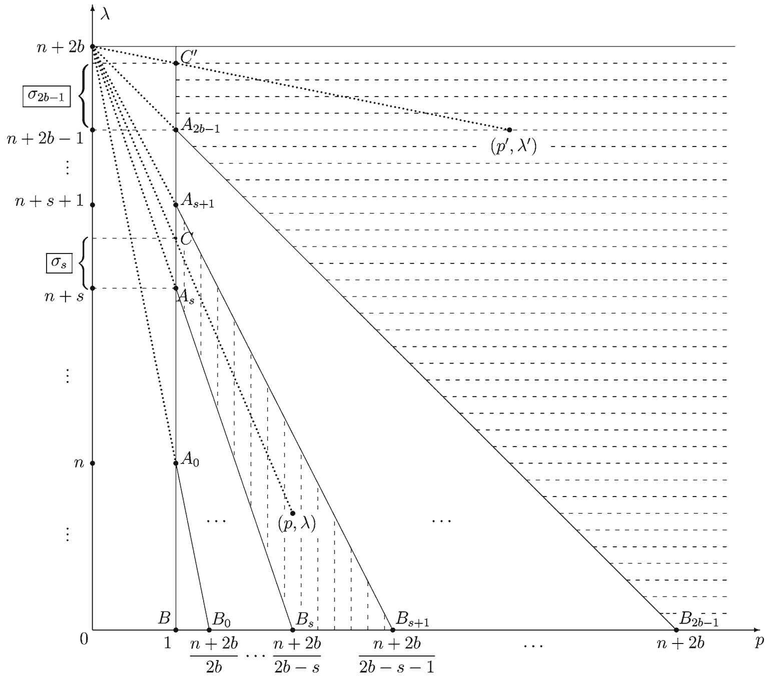

Fig. 1 The plane $O_{p,\lambda}$

whereas $(p, \lambda) \in (A_{s+1}, B_{s+1})$ implies $D^{s+1}\mathbf{u} \in BMO$. The situation is similar for the derivatives $D^{2b-1}\mathbf{u}$ as well (i.e., $s = 2b - 1$) but R_{2b-1} is now the shadowed *unbounded region* on the picture. Thus, $(p', \lambda') \in R_{2b-1}$ gives that $D^{2b-1}\mathbf{u}$ are Hölder continuous with exponent $\sigma_{2b-1} = |C'A_{2b-1}|$, $C' = C'(p', \lambda')$, while $(p', \lambda') \in (A_{2b-1}, B_{2b-1})$ yields $D^{2b-1}\mathbf{u} \in BMO$. To interpret the statement of Corollary 9, we simply have to consider points on the p-axis with $p > 1$. This way, $p > \frac{n+2b}{2b}$ implies $\mathbf{u} \in L^\infty(Q')$ whereas $D^s\mathbf{u}$ are Hölder continuous with exponent $\sigma_s = 2b - s - \frac{n+2b}{p}$ if $(p, 0) \in (B_s, B_{s+1})$ (with the setting $B_{2b} := +\infty$). We will derive in Sect. 4 an improvement of Corollary 9 which, loosely speaking, sounds like Corollary 11 with $\lambda = 0$. Thus (cf. Corollary 16), $D^s\mathbf{u} \in L^{p,(2b-s)p}(Q')$ if $(p, 0) \in (B, B_s)$ while $(p, 0) \equiv B_s$ gives $D^s\mathbf{u} \in VMO(Q')$.

3 Gaussian-type potentials

To obtain an explicit formula for the solution of the system (6), we fix a_α^{kj}'s at some interior point $(x_0, t_0) \in Q$, set $\mathbf{A}_\alpha^0 := \mathbf{A}_\alpha(x_0, t_0) = \left\{a_\alpha^{kj}(x_0, t_0)\right\}_{k,j=1}^m$, and consider the constant coefficients operator

$$\mathfrak{L}_0 := \mathbf{Id}_m D_t - \sum_{|\alpha|=2b} \mathbf{A}_\alpha^0 D^\alpha.$$

It is known (see [24]) that the fundamental matrix $\mathbf{\Gamma}_0(x,t) = \left\{\Gamma_0^{kj}(x,t)\right\}_{k,j=1}^m$ of $\mathfrak{L}_0$ has entries

$$\Gamma_0^{kj}(x,t) = L_{jk}(D_t, D^\alpha)\widetilde{\Gamma}_0(x,t),$$

where $\{L_{jk}(D_t,D^\alpha)\}_{j,k=1}^m$ is the *cofactor matrix* of $\{\mathbf{Id}_m D_t - \sum_{|\alpha|=2b} \mathbf{A}_\alpha^0 D^\alpha\} = \left\{\delta^{jk} D_t - \sum_{|\alpha|=2b} a_\alpha^{jk}(x_0,t_0)D^\alpha\right\}_{j,k=1}^m$, and $\widetilde{\Gamma}_0(x,t)$ is the fundamental solution of the *parabolic equation*

$$\det\left\{\mathbf{Id}_m D_t - \sum_{|\alpha|=2b} \mathbf{A}_\alpha^0 D^\alpha\right\}u = 0. \tag{12}$$

(Note that L_{jk} is either a homogeneous differential operator of order $2b(m-1)$ or the operator of multiplication by 0). Applying Fourier transform in x and Laplace transform in t, it is easy to get

$$\widetilde{\Gamma}_0(x,t) = \frac{1}{(2\pi)^n 2\pi i}\int_{\mathbb{R}^n} \mathrm{e}^{i(x\cdot\xi)}\mathrm{d}\xi \int_{\mathfrak{C}(\xi)} \frac{\mathrm{e}^{pt}}{\det\left\{p\,\mathbf{Id}_m - \sum_{|\alpha|=2b}\mathbf{A}_\alpha^0 (i\xi)^\alpha\right\}}\mathrm{d}p, \tag{13}$$

where $\mathfrak{C}(\xi)$ is a contour in the complex p-plane enclosing all the roots of (7) and therefore, in view of (8), could be taken to lie in the left half-plane. Thus (cf. [9]), the fundamental matrix $\mathbf{\Gamma}_0(x,t)$ of $\mathfrak{L}_0$ after a transformation of variables is given by

$$\mathbf{\Gamma}_0(x,t) = \begin{cases} \dfrac{1}{(2\pi)^n t^{n/2b}}\displaystyle\int_{\mathbb{R}^n} \exp\left\{i(xt^{-1/2b}\cdot\xi) + \sum_{|\alpha|=2b}\mathbf{A}_\alpha^0(i\xi)^\alpha\right\}d\xi & \text{for } t>0,\\ 0 & \text{for } t\le 0.\end{cases}$$

In other words, $\mathbf{\Gamma}_0(x,t)$ possesses properties analogous to these of the Gauss kernel (see [8,9,11,12,17,18,24]). Precisely,

($\mathcal{P}_1$) *Regularity:* $\mathbf{\Gamma}_0 \in C^\infty(\mathbb{R}^{n+1}\setminus\{0\})$.
($\mathcal{P}_2$) *Mixed homogeneity:* for any $\mu>0$ and any multiindex β it holds

$$\mathbf{\Gamma}_0(\mu x, \mu^{2b}t) = \mu^{-n}\mathbf{\Gamma}_0(x,t), \quad D^\beta\mathbf{\Gamma}_0(\mu x,\mu^{2b}t) = \mu^{-n-|\beta|}D^\beta\mathbf{\Gamma}_0(x,t).$$

($\mathcal{P}_3$) *Vanishing property on the unit sphere* $\mathbb{S}^n$:

$$\int_{\mathbb{S}^n} D^\alpha\mathbf{\Gamma}_0(\overline{x},\overline{t})d\sigma_{(\overline{x},\overline{t})} = \mathbf{0} \quad \text{for any } \alpha,\ |\alpha|=2b.$$

To show this property define the sets $A = \{(x,t):\ t>0,\ 1<\rho(x,t)<2\}$, $B=\{(x,t):\ 1<t<2^{2b}\}$, $B_k = \{(x,t):\ 0<t<k^{2b},\ \rho(x,t)>k\}$, $k=1,2$ (cf. [10, Appendix]). Straightforward calculations based on the properties of the metric, ($\mathcal{P}_2$) and $A = \big(B\cup B_1\big)\setminus B_2$ yield

$$\begin{aligned}\int_A D^\alpha\mathbf{\Gamma}_0(x,t)\mathrm{d}x\mathrm{d}t &= \int_A \big(\rho(x,t)\big)^{-n-2b} D^\alpha\mathbf{\Gamma}_0(\overline{x},\overline{t})\mathrm{d}x\mathrm{d}t\\ &= \int_1^2 \frac{d\rho}{\rho}\int_{\mathbb{S}^n} D^\alpha\mathbf{\Gamma}_0(\overline{x},\overline{t})\mathrm{d}\sigma_{(\overline{x},\overline{t})} = \log 2\int_{\mathbb{S}^n} D^\alpha\mathbf{\Gamma}_0(\overline{x},\overline{t})\mathrm{d}\sigma_{(\overline{x},\overline{t})}.\end{aligned}$$

On the other hand,

$$\int_{B_2} D^\alpha \mathbf{\Gamma}_0(x,t)\mathrm{d}x\mathrm{d}t = \frac{1}{2^{n+2b}}\int_{B_2} D^\alpha \mathbf{\Gamma}_0\left(\frac{x}{2},\frac{t}{2^{2b}}\right)\mathrm{d}x\mathrm{d}t = \int_{B_1} D^\alpha \mathbf{\Gamma}_0(x,t)\mathrm{d}x\mathrm{d}t,$$
$$\int_{B} D^\alpha \mathbf{\Gamma}_0(x,t)\mathrm{d}x\mathrm{d}t = \int_B \frac{1}{t^{\frac{n}{2b}+1}} D^\alpha \mathbf{\Gamma}_0\left(\frac{x}{t^{\frac{1}{2b}}},1\right)\mathrm{d}x\mathrm{d}t = \int_1^{2^{2b}}\frac{dt}{t}\int_{\mathbb{R}^n} D^\alpha \mathbf{\Gamma}_0(x,1)\mathrm{d}x$$

which gives

$$\begin{aligned}\int_A D^\alpha \mathbf{\Gamma}_0(x,t)\mathrm{d}x\mathrm{d}t =& \int_B \cdots + \int_{B_1}\cdots - \int_{B_2}\cdots = \int_B D^\alpha \mathbf{\Gamma}_0(x,t)\mathrm{d}x\mathrm{d}t\\ =& \ 2b\log 2\int_{\mathbb{R}^n} D^\alpha \mathbf{\Gamma}_0(x,1)\mathrm{d}x = \mathbf{0}.\end{aligned}$$

The last equality follows from the vanishing property on hyperplanes (cf. [24]) possessed by $\mathbf{\Gamma}_0(x,t)$ for any $t>0$:

$$\int_{\mathbb{R}^n} D_t^s D_x^\beta \mathbf{\Gamma}_0(x,t)dx = \begin{cases}\mathbf{Id}_m & \text{if } s+|\beta|=0,\\ \mathbf{0} & \text{if } s+|\beta|>0.\end{cases}$$

($\mathcal{P}_4$) *Boundedness of the derivatives:*

$$\sup_{(\overline{x},\overline{t})\in\mathbb{S}^n} |D^\beta \mathbf{\Gamma}_0(\overline{x},\overline{t})| \le C(n,\beta,\max_\alpha |\mathbf{A}^0_\alpha|) \quad \forall \text{ multiindex } \beta.$$

($\mathcal{P}_5$) *Integrability:*

$$D^\beta \mathbf{\Gamma}_0 \in L^1_{\mathrm{loc}}(\mathbb{R}^{n+1}) \ \text{ for } |\beta|<2b, \quad D^\alpha \mathbf{\Gamma}_0 \notin L^1_{\mathrm{loc}}(\mathbb{R}^{n+1}) \ \text{for } |\alpha|=2b.$$

Let $\mathbf{v}\in C^\infty(\mathbb{R}^{n+1})$ be compactly supported in x, $\mathbf{v}(x,0)=\mathbf{0}$. Take $(x_0,t_0)\in \operatorname{supp}\mathbf{v}$ and consider the system

$$\mathfrak{L}_0\mathbf{v} = \sum_{|\alpha|=2b}\big(\mathbf{A}_\alpha(x,t)-\mathbf{A}_\alpha(x_0,t_0)\big)D^\alpha\mathbf{v} + \mathfrak{L}\mathbf{v}(x,t) =: \mathbf{g}(x,t). \tag{14}$$

The solution $\mathbf{v}$ can be written as a Gaussian-type potential

$$\mathbf{v}(x,t) = \int_{\mathbb{R}^{n+1}} \mathbf{\Gamma}_0(x-y,t-\tau)\mathbf{g}(y,\tau)\mathrm{d}y\mathrm{d}\tau. \tag{15}$$

The higher order derivatives in x have the form

$$\begin{aligned}D^\alpha\mathbf{v}(x,t) =& \ P.V.\int_{\mathbb{R}^{n+1}} D^\alpha \mathbf{\Gamma}_0(x-y,t-\tau)\mathbf{g}(y,\tau)\mathrm{d}y\mathrm{d}\tau\\ &+\int_{\mathbb{S}^n} D^{\beta^s}\mathbf{\Gamma}_0(y,\tau)\nu_s d\sigma_{(y,\tau)}\,\mathbf{g}(x,t), \quad |\alpha|=2b,\end{aligned} \tag{16}$$

where $\mathbf{\Gamma}_0(x-y,t-\tau)=\mathbf{\Gamma}(x_0,t_0;x-y,t-\tau)$ and the dependence on (x_0,t_0) is through $\mathbf{A}^0_\alpha$. Since the choice of (x_0,t_0) is free, we set $(x_0,t_0)=(x,t)$, thus

$$\begin{aligned} D^\alpha \mathbf{v}(x,t) &= P.V.\int_{\mathbb{R}^{n+1}} D^\alpha \mathbf{\Gamma}(x,t;x-y,t-\tau)\mathfrak{L}\mathbf{v}(y,\tau)\mathrm{d}y\mathrm{d}\tau \\ &\quad + \sum_{|\alpha'|=2b} P.V.\int_{\mathbb{R}^{n+1}} D^\alpha \mathbf{\Gamma}(x,t;x-y,t-\tau) \\ &\quad \times\big(\mathbf{A}_{\alpha'}(y,\tau)-\mathbf{A}_{\alpha'}(x,t)\big)D_y^{\alpha'}\mathbf{v}(y,\tau)\mathrm{d}y\mathrm{d}\tau \\ &\quad + \int_{\mathbb{S}^n} D^{\beta^s}\mathbf{\Gamma}(x,t;y,\tau)\nu_s \mathrm{d}\sigma_{(y,\tau)}\,\mathfrak{L}\mathbf{v}(x,t) \\ &=: \mathfrak{K}_\alpha(\mathfrak{L}\mathbf{v}) + \sum_{|\alpha'|=2b}\mathfrak{C}_\alpha[\mathbf{A}_{\alpha'},D^{\alpha'}\mathbf{v}] + \mathbf{F}(x,t)\mathfrak{L}\mathbf{v}(x,t), \quad \forall\alpha: |\alpha|=2b \end{aligned} \tag{17}$$

where the derivatives $D^\alpha\mathbf{\Gamma}(\cdot,\cdot;\cdot,\cdot)$ are taken with respect to the third variable.

Denote $\mathbf{k}(x,t;y,\tau):=D_y^\alpha\mathbf{\Gamma}(x,t;y,\tau)$ with $|\alpha|=2b$. Each entry of the $m\times m$ matrix $\mathbf{k}$ is a Calderón-Zygmund kernel in the sense of the Definition 1. In fact, (i_a) and (i_b) are just properties $(\mathcal{P}_1)$ and $(\mathcal{P}_2)$ of the fundamental solution, while i_c) and ii) follow from $(\mathcal{P}_3)$ and $(\mathcal{P}_4)$. Finally, $(\mathcal{P}_5)$ shows that $\mathfrak{K}_\alpha$ and $\mathfrak{C}_\alpha$ are really singular integral operators. For what concerns their boundedness in Lebesgue and Morrey spaces, we have

Lemma 12 *Let $|\alpha|=|\alpha'|=2b$ and $\mathbf{A}_\alpha\in L^\infty(Q)$. For each $p\in(1,+\infty)$ there exists a constant $C=C(n,m,b,\delta,\|\mathbf{A}_\alpha\|_{\infty;Q},p)$ such that for any $\mathbf{f}\in L^p(Q)$*

$$\|\mathfrak{K}_\alpha\mathbf{f}\|_{p;Q}\le C\|\mathbf{f}\|_{p;Q},\quad \|\mathfrak{C}_\alpha[\mathbf{A}_{\alpha'},\mathbf{f}]\|_{p;Q}\le C\|\mathbf{A}_{\alpha'}\|_{*;Q}\|\mathbf{f}\|_{p;Q}. \tag{18}$$

For each $p\in(1,+\infty)$ and each $\lambda\in(0,n+2b)$ there is a constant C depending on n, m, b, δ, $\|\mathbf{A}_\alpha\|_{\infty;Q}$, p and λ such that for any $\mathbf{f}\in L^{p,\lambda}(Q)$

$$\|\mathfrak{K}_\alpha\mathbf{f}\|_{p,\lambda;Q}\le C\|\mathbf{f}\|_{p,\lambda;Q},\quad \|\mathfrak{C}_\alpha[\mathbf{A}_{\alpha'},\mathbf{f}]\|_{p,\lambda;Q}\le C\|\mathbf{A}_{\alpha'}\|_{*;Q}\|\mathbf{f}\|_{p,\lambda;Q}. \tag{19}$$

Moreover, let $\mathbf{A}_\alpha\in VMO(Q)\cap L^\infty(Q)$ with VMO-modulus $\eta_{\mathbf{A}_\alpha}$. Then for each $\varepsilon>0$ there exists $r_0=r_0(\varepsilon,\eta_{\mathbf{A}_\alpha})$ such that if $r<r_0$ we have

$$\|\mathfrak{C}_\alpha[\mathbf{A}_{\alpha'},\mathbf{f}]\|_{p;\mathcal{C}_r}\le C\varepsilon\|\mathbf{f}\|_{p;\mathcal{C}_r}\quad \forall\,\mathbf{f}\in L^p(\mathcal{C}_r), \tag{20}$$

$$\|\mathfrak{C}_\alpha[\mathbf{A}_{\alpha'},\mathbf{f}]\|_{p,\lambda;\mathcal{C}_r}\le C\varepsilon\|\mathbf{f}\|_{p,\lambda;\mathcal{C}_r}\quad \forall\,\mathbf{f}\in L^{p,\lambda}(\mathcal{C}_r) \tag{21}$$

for any parabolic cylinder $\mathcal{C}_r\subset\overline{Q}$.

The first bound in (18) is proved by Fabes and Riviére [10, Theorem 1] for general kernels of mixed homogeneity. The second one is obtained in [1, Theorem 2.12] in the case $b=1$. The passage from *constant* to *variable* kernels makes use of Calderón-Zygmund's approach [2,3] of expansion into spherical harmonics and leads to (18) for $b>1$ as well. The estimates (19) follow from Theorem 4 and Corollary 5 (cf. [19]). The estimates (20) and (21) follow from (18) and (19) on the base of $\mathbf{A}_\alpha\in VMO$ (see [6, Theorem 2.13],[19, Corollary 2.8], [23, Theorem 3.7]).

Remark 13 Employing density arguments and (18) it is easily seen that the representation formula (17) still holds true (almost everywhere) for *compactly supported in x functions* $\mathbf{v}\in W_p^{2b,1}$ such that $\mathbf{v}(x,0)=\mathbf{0}$.

4 Sketch of the proofs

Proof (Theorem 8) We shall prove (10) first, supposing $\mathbf{u} \in W^{2b,1}_{p,\mathrm{loc}}(Q)$. Without loss of generality, extend $\mathbf{u}(x,t)$ as $\mathbf{0}$ for $t < 0$, fix $(x_0, t_0) \in \operatorname{supp} \mathbf{u}$ and consider the parabolic cylinder $\mathcal{C}_r(x_0, t_0)$. Let $\mathbf{v} \in W^{2b,1}_p(\mathcal{C}_r)$ with $\mathbf{v}(x, t_0 - r^{2b}) = \mathbf{0}$. In view of Remark 13, (17) and Lemma 12, for each $\varepsilon > 0$ there is $r_0(\varepsilon, \eta_{\mathbf{A}_\alpha})$ such that

$$\|D^{2b}\mathbf{v}\|_{p;\mathcal{C}_r} \le C\left(\|\mathfrak{L}\mathbf{v}\|_{p;\mathcal{C}_r} + \varepsilon\|D^{2b}\mathbf{v}\|_{p;\mathcal{C}_r}\right)$$

whenever $r < r_0$. Choosing ε small enough we obtain

$$\|D^{2b}\mathbf{v}\|_{p;\mathcal{C}_r} \le C\|\mathfrak{L}\mathbf{v}\|_{p;\mathcal{C}_r}. \tag{22}$$

Let $r \in (0, r_0)$, $\theta \in (0, 1)$, $\theta' = \theta(3-\theta)/2 > \theta$ and define the cut-off function

$$\varphi(x,t) := \varphi_1(x)\varphi_2(t), \quad 0 \le \varphi \le 1, \tag{23}$$

with $\varphi_1 \in C_0^\infty(B_r(x_0))$ and $\varphi_2 \in C^\infty(\mathbb{R})$ such that

$$\varphi_1(x) = \begin{cases} 1 & x \in B_{\theta r}(x_0) \\ 0 & x \notin B_{\theta' r}(x_0) \end{cases} \qquad \varphi_2(t) = \begin{cases} 1 & t \in (t_0 - (\theta r)^{2b}, t_0] \\ 0 & t < t_0 - (\theta' r)^{2b}. \end{cases}$$

Since $\theta' - \theta = \theta(1-\theta)/2$, it is clear that $|D_x^s\varphi| \le C(s)[\theta(1-\theta)r]^{-s}$ for any $1 \le s \le 2b$ and $|D_t\varphi| \le C[\theta(1-\theta)r]^{-2b}$.

Defining $\mathbf{v} = \varphi\mathbf{u}$, calculating $\mathfrak{L}\mathbf{v} = (\mathfrak{L}v_1, \ldots, \mathfrak{L}v_m)^{\mathrm{T}}$, applying (22) to $\mathbf{v}$ and letting $\|\cdot\|_{p;\theta r} := \|\cdot\|_{p;\mathcal{C}_{\theta r}}$ for the sake of simplicity, we get into

$$\begin{aligned} \|D^{2b}\mathbf{u}\|_{p;\theta r} &\le \|D^{2b}\mathbf{v}\|_{p;\theta' r} \le C\|\mathfrak{L}\mathbf{v}\|_{p;\theta' r} \\ &\le C\left(\|\mathbf{f}\|_{p;\theta' r} + \sum_{s=1}^{2b-1} \frac{\|D^{2b-s}\mathbf{u}\|_{p;\theta' r}}{[\theta(1-\theta)r]^s} + \frac{\|\mathbf{u}\|_{p;\theta' r}}{[\theta(1-\theta)r]^{2b}}\right). \end{aligned}$$

Defining the seminorms

$$\Theta_s := \sup_{0<\theta<1} [\theta(1-\theta)r]^s \|D^s\mathbf{u}\|_{p;\theta r} \quad \forall\, s \in \{0, \ldots, 2b\},$$

the last inequality rewrites as

$$\Theta_{2b} \le C\left(r^{2b}\|\mathbf{f}\|_{p;r} + \sum_{s=1}^{2b-1} \Theta_s + \Theta_0\right). \tag{24}$$

Proposition 14 (Interpolation inequality) *There exists a constant C depending on n, m, b, p and s, but independent of r, and such that*

$$\Theta_s \le \varepsilon\Theta_{2b} + \frac{C}{\varepsilon^{s/(2b-s)}}\Theta_0 \quad \textit{for any } \varepsilon \in (0, 2).$$

Fixing $\theta = 1/2$, taking suitable $\varepsilon \in (0, 2)$ and interpolating the intermediate seminorms in (24), we obtain the following *Caccioppoli-type* estimate

$$\|D^{2b}\mathbf{u}\|_{p;r/2} \le C\left(\|\mathbf{f}\|_{p;r} + Cr^{-2b}\|\mathbf{u}\|_{p;r}\right) \tag{25}$$

which holds also for $\|D_t\mathbf{u}\|_{p;r/2}$ in view of the parabolic structure of (6). Therefore, the desired estimate (10) follows from (25) after choosing $r < \operatorname{dist}(\Omega', \partial\Omega'')$ and employing a finite covering of Q' by cylinders $\mathcal{C}_{r/2}$.

Turning back to Theorem 8, we use homotopy arguments in order to prove integrability property of $\mathfrak{L}$. For any couple of multiindices $\alpha, \alpha' \in \mathfrak{A} := \{\alpha \in \mathbb{N}^n : \ |\alpha| = 2b\}$ and any $\mathbf{g} \in L^\omega(\mathcal{C}_r)$, $\omega \in (1, \infty)$ define the operator $\mathbf{U}_{\alpha,\alpha'} : \ L^\omega(\mathcal{C}_r) \to L^\omega(\mathcal{C}_r)$ as

$$\mathbf{U}_{\alpha,\alpha'}\mathbf{g}(x,t) := P.V. \int_{\mathcal{C}_r} D^\alpha \mathbf{\Gamma}(x,t;x-y,t-\tau)\big(\mathbf{A}_{\alpha'}(y,\tau) - \mathbf{A}_{\alpha'}(x,t)\big)\mathbf{g}(y,\tau)\mathrm{d}y\mathrm{d}\tau. \tag{26}$$

Take the cut-off function (23) with $\theta = 1/2$ and set $\mathbf{v} = \varphi\mathbf{u}$. Then $\mathfrak{L}\mathbf{v} = \varphi(x,t)\mathfrak{L}\mathbf{u} + \mathfrak{L}'(x,t,D_x)\mathbf{u} \in L^{q_1}(\mathcal{C}_r)$ with $q_1 = \min\left\{p, \frac{q(n+2b)}{n+2b-q}\right\}$ if $q < n+2b$ and $q_1 = p$ otherwise (see [25, Theorem 2.5]). To show $\mathbf{v} \in W^{2b,1}_{q_1}(\mathcal{C}_r)$ we rely on formula (17) and define

$$\mathbf{V}_\alpha(x,t) := P.V. \int_{\mathcal{C}_r} D^\alpha \mathbf{\Gamma}(x,t;x-y,t-\tau)\mathfrak{L}\mathbf{v}(y,\tau)\mathrm{d}y\mathrm{d}\tau + \mathbf{F}(x,t)\mathfrak{L}\mathbf{v}(x,t) \tag{27}$$

which belongs to $L^{q_1}(\mathcal{C}_r)$ according to Lemma 12. Let $\omega \in [q, q_1]$ and define the operator $\mathbf{W}: \ (L^\omega(\mathcal{C}_r))^N \to (L^\omega(\mathcal{C}_r))^N$, $N = m\,\mathrm{card}\,(\mathfrak{A})$, by $\mathbf{W}(\mathbf{w}) = (\mathbf{W}_\alpha(\mathbf{w}))_{\alpha\in\mathfrak{A}}$ where

$$\mathbf{W}_\alpha(\mathbf{w}) := \mathbf{V}_\alpha + \sum_{\alpha'\in\mathfrak{A}} \mathbf{U}_{\alpha,\alpha'}(\mathbf{w}_{\alpha'}). \tag{28}$$

If r is small enough then $\sum_{\alpha,\ \alpha'\in\mathfrak{A}} \|\mathbf{U}_{\alpha,\alpha'}\|_{L^\omega(\mathcal{C}_r)} < 1$ because of $\mathbf{A}_\alpha \in VMO$ and (20) and $\mathbf{W}$ is a contraction mapping. Hence, there is a unique fixed point $\mathbf{w}_0 = \mathbf{W}(\mathbf{w}_0)$ belonging to any $(L^\omega(\mathcal{C}_r))^N$ for each $\omega \in [q, q_1]$.

Whence $D^{2b}\mathbf{v} \in L^{q_1}(\mathcal{C}_r)$ according to (17) and this implies $D^{2b}\mathbf{u}$, $D_t\mathbf{u} \in L^{q_1}(\mathcal{C}_{r/2})$ by virtue of $\mathbf{f} \in L^p_{\mathrm{loc}}(Q)$ and the parabolic structure of (6). To get $\mathbf{u} \in W^{2b,1}_{p,\mathrm{loc}}(Q)$, it remains to iterate the above procedure finitely many times until $q_1 = p$. □

Proof (Theorem 10) Let $\mathbf{u} \in W^{2b,1}_{p,\lambda,\mathrm{loc}}(Q)$. Then $\mathbf{u} \in W^{2b,1}_{p,\mathrm{loc}}(Q)$ and the representation formula (17) applied to $\mathbf{v} = \varphi\mathbf{u}$ still holds true. This way the estimate (11) follows as in the preceding proof making use of (19) and (21).

We shall stress now our attention on the improving-of-integrability property of the operator $\mathfrak{L}$ in Morrey spaces. That is, let $\mathbf{f} \in L^{p,\lambda}_{\mathrm{loc}}(Q)$ and suppose $\mathbf{u} \in W^{2b,1}_{q,\mathrm{loc}}(Q)$ is a solution of (6) with $1 < q \le p < +\infty$. Since $\mathbf{f} \in L^p_{\mathrm{loc}}(Q)$, Theorem 8 gives $\mathbf{u} \in W^{2b,1}_{p,\mathrm{loc}}(Q)$. Take arbitrary $\omega \in (1, +\infty)$ and $\mu \in (0, n+2b)$. In view of (19), the operator $\mathbf{U}_{\alpha,\alpha'}$ as given by (26), is well-defined from $L^{\omega,\mu}(\mathcal{C}_r)$ into itself and its norm is less than 1 if r is small enough. Arguing as in the preceding proof, we get $\mathfrak{L}\mathbf{v} = \varphi(x,t)\mathfrak{L}\mathbf{u} + \mathfrak{L}'(x,t,D_x)\mathbf{u}$ with $\mathrm{ord}\,\mathfrak{L}' = 2b-1$. By hypothesis $\mathfrak{L}\mathbf{u} \in L^{p,\lambda}(\mathcal{C}_r)$ and $\mathfrak{L}'\mathbf{u} \in L^p(\mathcal{C}_r)$. Employing Hölder's inequality, we get $\mathfrak{L}'\mathbf{u} \in L^{p,\lambda_1}(\mathcal{C}_r)$ with $\lambda_1 = \min\{p, \lambda\}$ and therefore $\mathfrak{L}\mathbf{v} \in L^{p,\lambda_1}(\mathcal{C}_r)$ as well. To get $\mathbf{v} \in W^{2b,1}_{p,\lambda_1}(\mathcal{C}_r)$, we use (27) to define $\mathbf{V}_\alpha$ which is in $L^{p,\lambda_1}(\mathcal{C}_r)$ as Lemma 12 asserts. Taking $\mu \in [0, \lambda_1]$ and consider the operator $\mathbf{W}: \ (L^{p,\mu}(\mathcal{C}_r))^N \to (L^{p,\mu}(\mathcal{C}_r))^N$ as given by (28), it follows as in the proof of Theorem 8 that $\mathbf{W}$ possesses a unique fixed point $\mathbf{w}_0 = (D^\alpha\mathbf{v})_{\alpha\in\mathfrak{A}} \in (L^{p,\mu}(\mathcal{C}_r))^N$ for all $\mu \in [0, \lambda_1]$. If $\lambda_1 = \lambda$ we have $D^{2b}\mathbf{u},\ D_t\mathbf{u} \in L^{p,\lambda}(\mathcal{C}_{r/2})$ as desired. Otherwise, $\lambda_1 = p < \lambda$ and $\mathbf{u} \in W^{2b,1}_{p,p}(\mathcal{C}_{r/2})$. Take again the cut-off function φ (cf. (23)) with $\theta = 1/4$ and calculate $\mathfrak{L}\mathbf{v}$ as above. Thus, $\mathfrak{L}\mathbf{v} \in L^{p,\lambda_2}(\mathcal{C}_{r/2})$ with $\lambda_2 = \min\{2p, \lambda\}$ (see Lemma 15) and the arguments already used give $D^{2b}\mathbf{u},\ D_t\mathbf{u} \in L^{p,\lambda_2}(\mathcal{C}_{r/4})$. It remains to repeat the procedure finitely many times in order to complete the proof of Theorem 10. □

Proof (Corollary 11) To obtain precise regularity of the lower order derivatives we need of the following Poincaré type inequality. □

Lemma 15 *Let* $\mathbf{u} \in W_p^{2b,1}(\mathcal{C}_r)$*, then, for each* $s \in \{0, 1, \ldots, 2b-1\}$*, we have*

$$\int_{\mathcal{C}_r} |D^s\mathbf{u}(x,t) - (D^s\mathbf{u})_{\mathcal{C}_r}|^p \mathrm{d}x\mathrm{d}t \\ \leq C\left(r^{(2b-s)p}\left(\|D^{2b}\mathbf{u}\|^p_{p;\mathcal{C}_r} + \|D_t\mathbf{u}\|^p_{p;\mathcal{C}_r}\right) + r^p\|D^{s+1}\mathbf{u}\|^p_{p;\mathcal{C}_r}\right)$$

where $(D^s\mathbf{u})_{\mathcal{C}_r}$ *is the integral average of* $D^s\mathbf{u}$ *over* $\mathcal{C}_r$ *and* $C = C(p, m, n, s)$*.*

Let $Q_r = \mathcal{C}_r \cap Q'$ where $2r < \operatorname{dist}(\Omega', \partial\Omega'')$. To begin with, take $s = 2b - 1$. Direct calculations based on Lemma 15 lead to

$$\begin{aligned} &\frac{1}{r^{p+\lambda}} \int_{Q_r} |D^{2b-1}\mathbf{u}(x,t) - (D^{2b-1}\mathbf{u})_{Q_r}|^p \mathrm{d}x\mathrm{d}t \\ &\leq C(n,p,m)\left(\frac{1}{r^\lambda}\|D^{2b}\mathbf{u}\|^p_{p;Q_r} + \frac{1}{r^\lambda}\|D_t\mathbf{u}\|^p_{p;Q_r}\right) \\ &\leq C\left(\|D^{2b}\mathbf{u}\|^p_{p,\lambda;\widetilde{Q}'} + \|D_t\mathbf{u}\|^p_{p,\lambda;\widetilde{Q}'}\right), \end{aligned} \tag{29}$$

where $\widetilde{Q}' = \widetilde{\Omega}' \times (0, T)$ and $\Omega' \Subset \widetilde{\Omega}' \Subset \Omega''$. Taking the supremum with respect to r we get *the Campanato seminorm* of $D^{2b-1}\mathbf{u}$ on the left-hand side which, in view of (11), turns out to be bounded by the Morrey norms of $\mathbf{f}$ and $\mathbf{u}$ in Q''. Now, employing the embedding properties of Campanato spaces into Morrey and Hölder ones (see [4], and [16] for more general functional settings) we obtain as follows. If $p+\lambda < n+2b$ then $D^{2b-1}\mathbf{u} \in L^{p,p+\lambda}(Q')$ and $\|D^{2b-1}\mathbf{u}\|_{p,p+\lambda;Q'}$ is controlled in terms of $\|\mathbf{f}\|_{p,\lambda;Q''}$ and $\|\mathbf{u}\|_{p,\lambda;Q''}$. If $p+\lambda > n+2b$ then $D^{2b-1}\mathbf{u} \in C^{\sigma_{2b-1},\sigma_{2b-1}/2b}(Q')$ with $\sigma_{2b-1} = 1 - \frac{n+2b-\lambda}{p}$ (cf. [16, Corollary 1]). Finally, if $p + \lambda = n + 2b$ we first apply the Hölder inequality to $\|D^{2b-1}\mathbf{u} - (D^{2b-1}\mathbf{u})_{\mathcal{C}_r}\|_{1;Q_r}$ and then Lemma 15 in order to obtain $D^{2b-1}\mathbf{u} \in BMO(Q')$. Having in mind the recursive character of Lemma 15, we shall complete the proof of Corollary 11 by running induction for decreasing values of s (see [20] for details).

The method employed in the proof of Corollary 11 could be applied also to derive Corollary 9 directly, without relying on Besov spaces. Moreover, repeating the arguments already used with $\lambda = 0$, we get the following refinement of Corollary 9.

Corollary 16 *Under the hypotheses of Theorem 8, fix an* $s \in \{0, 1, \ldots, 2b-1\}$*. Then there is a constant* C *such that*

- *if* $p \in \left(1, \frac{n+2b}{2b-s}\right)$ *then* $D^s\mathbf{u} \in L^{p,(2b-s)p}(Q')$ *and*

$$\|D^s\mathbf{u}\|_{p,(2b-s)p;Q'} \leq C\left(\|\mathbf{f}\|_{p;Q''} + \|\mathbf{u}\|_{p;Q''}\right);$$

- *if* $p = \frac{n+2b}{2b-s}$ *then* $D^s\mathbf{u} \in BMO(Q')$*,*

$$\|D^s\mathbf{u}\|_{*;Q'} \leq C\left(\|\mathbf{f}\|_{p;Q''} + \|\mathbf{u}\|_{p;Q''}\right)$$

  and moreover, $D^s\mathbf{u} \in VMO(Q')$*.*

5 Interior regularity for elliptic systems

Consider a linear elliptic system in an open bounded domain Ω in $\mathbb{R}^n$, $n \geq 2$

$$\mathfrak{L}(x, D)\mathbf{u} := \sum_{|\alpha|=2b} \mathbf{A}_\alpha(x) D^\alpha \mathbf{u}(x) = \mathbf{f}(x) \tag{30}$$

for the unknown vector-valued function $\mathbf{u}$: $\Omega \to \mathbb{R}^m$ where $\mathbf{A}_\alpha(x)$ is the $m \times m$-matrix $\left\{a_\alpha^{jk}(x)\right\}_{j,k=1}^m$ and $a_\alpha^{jk}: \Omega \to \mathbb{R}$ are measurable functions. We suppose (30) to be an *elliptic system,* that is, the characteristic determinant of $\mathfrak{L}(x, \xi)$ is non-vanishing for a.a. $x \in \Omega$ and all $\xi \neq 0$. This rewrites as (see [5,7])

$$\exists\, \delta > 0: \quad \det \left\{ \sum_{|\alpha|=2b} \mathbf{A}_\alpha(x)\xi^\alpha \right\} \geq \delta|\xi|^{2bm} \quad \text{a.a. } x \in \Omega,\ \forall \xi \in \mathbb{R}^n. \tag{31}$$

Our goal is to obtain interior Hölder regularity of the solutions to (30) as a byproduct of the following a'priori estimate in Sobolev–Morrey classes.

Theorem 17 *Suppose* (31)*,* $a_\alpha^{jk} \in VMO(\Omega) \cap L^\infty(\Omega)$, $\mathbf{f} \in L^{p,\lambda}_{\text{loc}}(\Omega)$, $1 < p < \infty$, $0 < \lambda < n$, *and let* $\mathbf{u} \in W^{2b,p,\lambda}_{\text{loc}}(\Omega)$ *be a strong solution of* (30)*. Then, for any* $\Omega' \Subset \Omega'' \Subset \Omega$ *there is a constant C depending on n, m, b, p,* λ*,* δ*,* $\|a_\alpha^{jk}\|_{\infty;\Omega}$*, the* VMO*-moduli* $\eta_{a_\alpha^{jk}}$ *of the coefficients* (cf. [5,6]) *and* dist $(\Omega', \partial\Omega'')$*, such that*

$$\|\mathbf{u}\|_{W^{2b,p,\lambda}(\Omega')} \leq C \left(\|\mathbf{f}\|_{p,\lambda;\Omega''} + \|\mathbf{u}\|_{p,\lambda;\Omega''} \right). \tag{32}$$

It turns out, moreover, that the operator $\mathfrak{L}$ improves the integrability of solutions to (30). In fact, by means of standard homotopy arguments and making use of formula (17) (cf. [5,19,21, Sect. 3]), it is easy to get

Corollary 18 *Under the hypotheses of Theorem* 17*, suppose* $\mathbf{u} \in W^{2b,q}_{\text{loc}}(\Omega)$ *with* $q \in (1, p]$*. Then* $\mathbf{u} \in W^{2b,p,\lambda}_{\text{loc}}(\Omega)$.

A combination of (32) with the embedding properties of Sobolev–Morrey spaces leads to a precise characterization of the Morrey, BMO and Hölder regularity of the solutions to (30).

Corollary 19 *Under the hypotheses of Theorem* 17 *define* s_0 *as the least non-negative integer such that* $\frac{n}{2b-s_0} > 1$ *and fix an* $s \in \{s_0, \dots, 2b-1\}$*. Then there is a constant C such that:*

- *if* $p \in \left(1, \frac{n-\lambda}{2b-s}\right)$ *then* $D^s\mathbf{u} \in L^{p,(2b-s)p+\lambda}(\Omega')$ *and*

$$\|D^s\mathbf{u}\|_{p,(2b-s)p+\lambda;\Omega'} \leq C \left(\|\mathbf{f}\|_{p,\lambda;\Omega''} + \|\mathbf{u}\|_{p,\lambda;\Omega''} \right);$$

- *if* $p = \frac{n-\lambda}{2b-s}$ *then* $D^s\mathbf{u} \in BMO(\Omega')$ *and*

$$\|D^s\mathbf{u}\|_{BMO;\Omega'} \leq C \left(\|\mathbf{f}\|_{p,\lambda;\Omega''} + \|\mathbf{u}\|_{p,\lambda;\Omega''} \right);$$

- *if* $p \in \left(\frac{n-\lambda}{2b-s}, \frac{n-\lambda}{2b-s-1}\right)$[2] *then* $D^s\mathbf{u} \in C^{0,\sigma_s}(\Omega')$ *with* $\sigma_s = 2b - s - \frac{n-\lambda}{p}$ *and*

$$\sup_{\substack{x \neq x' \\ x,\, x' \in \Omega'}} \frac{\left|D^s\mathbf{u}(x) - D^s\mathbf{u}(x')\right|}{|x - x'|^{\sigma_s}} \leq C \left(\|\mathbf{f}\|_{p,\lambda;\Omega''} + \|\mathbf{u}\|_{p,\lambda;\Omega''} \right).$$

[2] This rewrites as $p \in (n - \lambda, \infty)$ when $s = 2b - 1$.

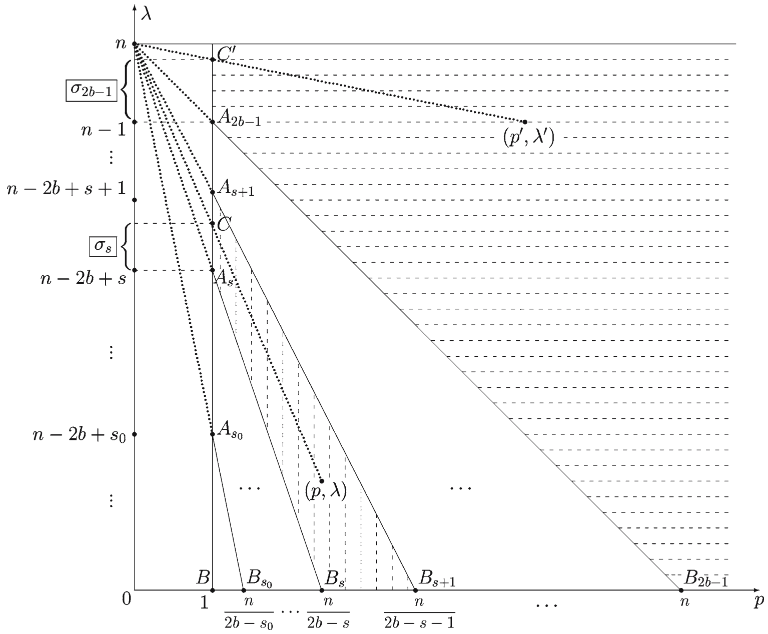

Fig. 2 The plane $O_{p,\lambda}$

If $s_0 \geq 1$ (*i.e.*, $2b \geq n$) *and* $p \in \left(1, \frac{n-\lambda}{2b-s_0}\right)$ *then* $\mathbf{u} \in C^{s_0-1, 2b-s_0+1-\frac{n-\lambda}{p}}(\Omega')$ *and*

$$\sup_{\substack{x \neq x' \\ x, x' \in \Omega'}} \frac{\left|D^{s_0-1}\mathbf{u}(x) - D^{s_0-1}\mathbf{u}(x')\right|}{|x - x'|^{2b-s_0+1-\frac{n-\lambda}{p}}} \leq C \left(\|\mathbf{f}\|_{p,\lambda;\Omega''} + \|\mathbf{u}\|_{p,\lambda;\Omega''}\right).$$

Figure 2 illustrates geometrically the results of Corollary 19 with the couple (p, λ) lying in the semistrip $\{(p, \lambda) \colon\ p > 1,\ 0 < \lambda < n\}$ and $s \in \{s_0, \ldots, 2b-1\}$. The points B_s on the p-axis are $B_s = \left(\frac{n}{2b-s}, 0\right)$, $B = (1, 0)$, and $A_s = (1, n-2b+s)$ is the intersection of the line through $(0, n)$ and B_s with the vertical line $\{p = 1\}$.

When (p, λ) belongs to the open right triangle $\triangle B B_s A_s$ then $D^s\mathbf{u} \in L^{p,(2b-s)p+\lambda}(\Omega')$. If (p, λ) lies on the open segment (A_s, B_s) we have $D^s\mathbf{u} \in BMO(\Omega')$. In particular, $(p, \lambda) \in \triangle B B_{s_0} A_{s_0}$ yields $D^{s_0}\mathbf{u} \in L^{p,(2b-s_0)p+\lambda}(\Omega')$, while $\mathbf{u} \in C^{s_0-1, 2b-s_0+1-\frac{n-\lambda}{p}}(\Omega')$ if $s_0 \geq 1$. Further on, $(p, \lambda) \in (A_{s_0}, B_{s_0})$ gives $D^{s_0}\mathbf{u} \in BMO(\Omega')$.

Let $s \in \{s_0, \ldots, 2b-2\}$ and suppose (p, λ) lies in the interior of the quadrilateral $Q_s := B_s B_{s+1} A_{s+1} A_s$. Then $D^s\mathbf{u} \in C^{0,\sigma_s}(\Omega')$ whereas $D^{s+1}\mathbf{u} \in L^{p,(2b-s-1)p+\lambda}(\Omega')$. Moreover, the exponent σ_s is the length of the segment (C, A_s) where $C = C(p, \lambda)$ is the intersection of the line $\{p = 1\}$ with the line passing through the points (p, λ) and $(0, n)$. In particular, $(p, \lambda) \in (A_{s+1}, B_{s+1})$ implies $D^{s+1}\mathbf{u} \in BMO(\Omega')$.

Similarly, set Q_{2b-1} for the shadowed *unbounded region* on the picture. Then $(p', \lambda') \in Q_{2b-1}$ gives $D^{2b-1}\mathbf{u} \in C^{0,\sigma_{2b-1}}(\Omega')$ with σ_{2b-1} equals to the length $|C' A_{2b-1}|$, $C' = C'(p', \lambda')$, while $D^{2b-1}\mathbf{u} \in BMO(\Omega')$ if $(p', \lambda') \in (A_{2b-1}, B_{2b-1})$.

6 Newtonian-type potentials

Fix the coefficients of (30) at a point $x_0 \in \Omega$ and consider the constant coefficients operator $\mathfrak{L}(x_0, D) := \sum_{|\alpha|=2b} \mathbf{A}_\alpha(x_0) D^\alpha$. Let $\boldsymbol{\Gamma}(x_0; x)$ be the fundamental matrix of $\mathfrak{L}(x_0, D)$ (cf. [7,24]).

Take $r > 0$ so small that $B_r = \{x \in \mathbb{R}^n : |x - x_0| < r\} \Subset \Omega$, and let $\mathbf{v} \in C_0^\infty(B_r)$. Then, employing

$$\mathfrak{L}(x_0, D)\mathbf{v}(x) = \big(\mathfrak{L}(x_0, D) - \mathfrak{L}(x, D)\big)\mathbf{v}(x) + \mathfrak{L}(x, D)\mathbf{v}(x)$$

and using standard approach (cf. [5,7,18]), we obtain a representation of $\mathbf{v}$ in terms of the Newtonian-type potentials

$$\mathbf{v}(x) = \int_{B_r} \boldsymbol{\Gamma}(x_0; x - y)\mathfrak{L}\mathbf{v}(y)\mathrm{d}y + \int_{B_r} \boldsymbol{\Gamma}(x_0; x - y)\big(\mathfrak{L}(x_0, D) - \mathfrak{L}(y, D)\big)\mathbf{v}(y)\mathrm{d}y.$$

Taking the $2b$-order derivatives and then unfreezing the coefficients by putting $x_0 = x$, we get

$$\begin{aligned} D^\alpha \mathbf{v}(x) = {} & \underbrace{P.V. \int_{B_r} D^\alpha \boldsymbol{\Gamma}(x; x - y)\mathfrak{L}\mathbf{v}(y)\mathrm{d}y}_{=:\, \mathfrak{K}_\alpha(\mathfrak{L}\mathbf{v})} \\ & + \sum_{|\alpha'|=2b} \underbrace{p.v. \int_{B_r} D^\alpha \boldsymbol{\Gamma}(x; x - y)\big(\mathbf{A}_{\alpha'}(x) - \mathbf{A}_{\alpha'}(y)\big) D^{\alpha'}\mathbf{v}(y)\mathrm{d}y}_{=:\, \mathfrak{C}_\alpha[\mathbf{A}_{\alpha'}, D^{\alpha'}\mathbf{v}]} \\ & + \int_{\mathbb{S}^{n-1}} D^{\beta^s} \boldsymbol{\Gamma}(x; y)\nu_s \mathrm{d}\sigma_y\, \mathfrak{L}\mathbf{v}(x) \quad \forall\, \alpha : |\alpha| = 2b \end{aligned} \tag{33}$$

where the derivatives $D^\alpha \boldsymbol{\Gamma}(\cdot;\cdot)$ are taken with respect to the second variable, the multiindices β^s are such that $\beta^s := (\alpha_1, \ldots, \alpha_{s-1}, \alpha_s - 1, \alpha_{s+1}, \ldots, \alpha_n)$, $|\beta^s| = 2b - 1$ and ν_s is the s-th component of the outer normal to $\mathbb{S}^{n-1}$.

Noting that each entry of the matrix $D^\alpha \boldsymbol{\Gamma}(x; y)$, $|\alpha| = 2b$, is a *Calderón–Zygmund kernel* (cf. [5,6]), we have

Lemma 20 *Let $|\alpha| = |\alpha'| = 2b$ and $\mathbf{A}_\alpha \in VMO \cap L^\infty(\Omega)$ with VMO-modulus $\eta_{\mathbf{A}_\alpha}$. For each $p \in (1, \infty)$ and each $\lambda \in (0, n)$ there is a constant C depending on n, m, b, δ, $\|\mathbf{A}_\alpha\|_{\infty;\Omega}$, p and λ such that*

$$\|\mathfrak{K}_\alpha \mathbf{f}\|_{p,\lambda;\Omega} \le C\|\mathbf{f}\|_{p,\lambda;\Omega}, \quad \|\mathfrak{C}_\alpha[\mathbf{A}_{\alpha'}, \mathbf{f}]\|_{p,\lambda;\Omega} \le C\|\mathbf{A}_{\alpha'}\|_{*;\Omega}\|\mathbf{f}\|_{p,\lambda;\Omega} \tag{34}$$

for all $\mathbf{f} \in L^{p,\lambda}(\Omega)$. Moreover, for each $\varepsilon > 0$ there exists $r_0 = r_0(\varepsilon, \eta_{\mathbf{A}_\alpha})$ such that if $r < r_0$ we have

$$\|\mathfrak{C}_\alpha[\mathbf{A}_{\alpha'}, \mathbf{f}]\|_{p,\lambda;B_r} \le C\varepsilon\|\mathbf{f}\|_{p,\lambda;B_r} \tag{35}$$

for all $B_r \Subset \Omega$ and all $\mathbf{f} \in L^{p,\lambda}(B_r)$.

Lemma 20 is a particular case of Corollaries 5 and 6 (see [19,21]).

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